Solutions to short-answer questions

1 a $$\begin{aligned}2z_1+3z_2&=2m+2ni+3p+3qi\\&=(2m+3p)+(2n+3q)i\end{aligned}$$

b $\overline{z}_2=p-qi$

c $$\begin{aligned}z_1\,\overline{z}_2&=(m+ni)(p-qi)\\&=mp+npi-mqi-nqi^2\\&=(mp+nq)+(np-mq)i\end{aligned}$$

d $$\begin{aligned}\frac{z_1}{z_2}&=\frac{m+ni}{p+qi}\\&=\frac{m+ni}{p+qi}\times\frac{p-qi}{p-qi}\\&=\frac{mp+npi-mqi-nqi^2}{p^2+q^2}\\&=\frac{(mp+nq)+(np-mq)i}{p^2+q^2}\end{aligned}$$

e $$\begin{aligned}z_1+\overline{z}_1&=(m+ni)+(m-ni)\\&=2m\end{aligned}$$

f $$\begin{aligned}(z_1+z_2)(z_1-z_2)&=z_1^2-z_2^2\\&=m^2+2mni+n^2i^2-(p^2+2pqi+q^2i^2)\\&=m^2+2mni-n^2-(p^2+2pqi-q^2)\\&=(m^2-n^2-p^2+q^2)+(2mn-2pq)i\end{aligned}$$

g $$\begin{aligned}\frac{1}{z_1}&=\frac{1}{m+ni}\\&=\frac{1}{m+ni}\times\frac{m-ni}{m-ni}\\&=\frac{m-ni}{m^2+n^2}\end{aligned}$$

h $$\begin{aligned}\frac{z_2}{z_1}&=\frac{p+qi}{m+ni}\\&=\frac{p+qi}{m+ni}\times\frac{m-ni}{m-ni}\\&=\frac{mp+nq+(mq-np)i}{m^2+n^2}\end{aligned}$$

i $$\begin{aligned}\frac{3z_1}{z_2}&=\frac{3(m+ni)}{p+qi}\\&=\frac{3(m+ni)}{p+qi}\times\frac{p-qi}{p-qi}\\&=\frac{3(mp+npi-mqi-nqi^2)}{p^2+q^2}\\&=\frac{3[(mp+nq)+(np-mq)i]}{p^2+q^2}\end{aligned}$$

2 a $A: z=1-\sqrt{3}i$

b $$\begin{aligned}B: z^2&=(1-\sqrt{3}i)^2\\&=1-2\sqrt{3}i+3i^2\\&=-2-2\sqrt{3}i\end{aligned}$$

c $C: z^3 = z^2 \times z$

$$= (-2 - 2\sqrt{3}i)(1 - \sqrt{3}i)$$
$$= -2 + 2\sqrt{3}i - 2\sqrt{3}i + 6i^2$$
$$= -8$$

d $D: \dfrac{1}{z} = \dfrac{1}{1 - \sqrt{3}i}$

$$= \frac{1}{1 - \sqrt{3}i} \times \frac{1 + \sqrt{3}i}{1 + \sqrt{3}i}$$
$$= \frac{1 + \sqrt{3}i}{4}$$

e $E: \bar{z} = 1 + \sqrt{3}i$

f $F: \dfrac{1}{\bar{z}} = \dfrac{1}{1 + \sqrt{3}i}$

$$= \frac{1}{1 + \sqrt{3}i} \times \frac{1 - \sqrt{3}i}{1 - \sqrt{3}i}$$
$$= \frac{1 - \sqrt{3}i}{4}$$

Note: use existing diagram from answers

3 a The point is in the first quadrant.

$$r = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\theta = \frac{\pi}{4}$$
$$\therefore \quad 1 + i = \sqrt{2} \text{ cis} \left(\frac{\pi}{4}\right)$$

b The point is in the fourth quadrant.

$$r = \sqrt{1 + 3}$$
$$= 2$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = -\frac{\pi}{3}$$
$$\therefore \quad 1 - \sqrt{3}i = 2 \text{ cis} \left(-\frac{\pi}{3}\right)$$

c The point is in the first quadrant.

$$r = \sqrt{12 + 1}$$
$$= \sqrt{13}$$
$$\tan\theta = \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{6}$$
$$\therefore \quad 2\sqrt{3} + i = \sqrt{13} \text{ cis} \left(\tan^{-1} \frac{\sqrt{3}}{6}\right)$$

d The point is in the first quadrant.

$$\begin{aligned} r &= \sqrt{18+18} \\ &= \sqrt{36} = 6 \\ \cos\theta &= \frac{3\sqrt{2}}{6} = \frac{1}{\sqrt{2}} \\ \theta &= \frac{\pi}{4} \end{aligned}$$

$$\therefore \quad 3\sqrt{2} + 3\sqrt{2}i = 6\text{ cis}\left(\frac{\pi}{4}\right)$$

e The point is in the third quadrant.

$$\begin{aligned} r &= \sqrt{18+18} \\ &= \sqrt{36} = 6 \\ \cos\theta &= -\frac{3\sqrt{2}}{6} = -\frac{1}{\sqrt{2}} \\ \theta &= -\pi + \frac{\pi}{4} = -\frac{3\pi}{4} \end{aligned}$$

$$\therefore \quad -3\sqrt{2} - 3\sqrt{2}i = 6\text{ cis}\left(-\frac{3\pi}{4}\right)$$

f The point is in the fourth quadrant.

$$\begin{aligned} r &= \sqrt{3+1} \\ &= 2 \\ \cos\theta &= \frac{\sqrt{3}}{2} \\ \theta &= -\frac{\pi}{6} \end{aligned}$$

$$\therefore \quad \sqrt{3} - i = 2\text{ cis}\left(-\frac{\pi}{6}\right)$$

4 a

$$\begin{aligned} x &= -2\cos\left(\frac{\pi}{3}\right) \\ &= -1 \\ y &= -2\sin\left(\frac{\pi}{3}\right) \\ &= -\sqrt{3} \end{aligned}$$

$$\therefore \quad z = -1 - \sqrt{3}i$$

b

$$\begin{aligned} x &= 3\cos\left(\frac{\pi}{4}\right) \\ &= \frac{3\sqrt{2}}{2} \\ y &= 3\sin\left(\frac{\pi}{4}\right) \\ &= \frac{3\sqrt{2}}{2} \end{aligned}$$

$$\therefore \quad z = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

c

$$\begin{aligned} x &= 3\cos\left(\frac{3\pi}{4}\right) \\ &= -\frac{3\sqrt{2}}{2} \\ y &= 3\sin\left(\frac{3\pi}{4}\right) \\ &= \frac{3\sqrt{2}}{2} \end{aligned}$$

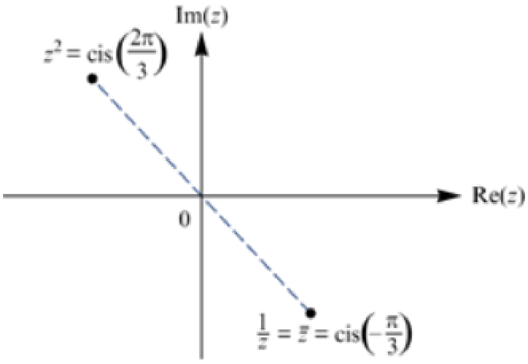
$$\therefore \quad z = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

d $$x = -3\cos\left(-\frac{3\pi}{4}\right)$$
$$= \frac{3\sqrt{2}}{2}$$
$$y = -3\sin\left(-\frac{3\pi}{4}\right)$$
$$= \frac{3\sqrt{2}}{2}$$
$$\therefore \quad z = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

e $$x = 3\cos\left(-\frac{5\pi}{6}\right)$$
$$= -\frac{3\sqrt{3}}{2}$$
$$y = 3\sin\left(-\frac{5\pi}{6}\right)$$
$$= -\frac{3}{2}$$
$$\therefore \quad z = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

f $$x = \sqrt{2}\cos\left(-\frac{\pi}{4}\right)$$
$$= 1$$
$$y = \sqrt{2}\ \sin\left(-\frac{\pi}{4}\right)$$
$$= -1$$
$$\therefore \quad z = 1 - i$$

5



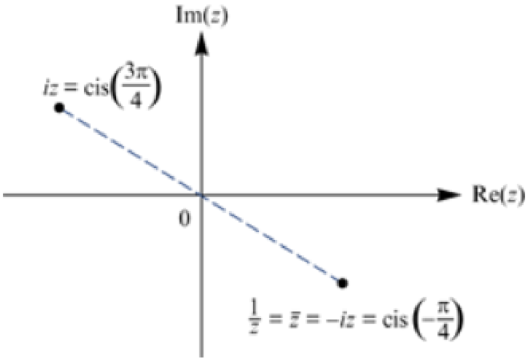
a $z^2 = \text{cis}\left(\frac{2\pi}{3}\right)$

b $\bar{z} = \text{cis}\left(-\frac{\pi}{3}\right)$

c $\frac{1}{z} = \text{cis}\left(-\frac{\pi}{3}\right)$

d $\text{cis}\left(\frac{2\pi}{3}\right)$

6



a $iz = \text{cis}\left(\frac{3\pi}{4}\right)$

b $\overline{z} = \text{cis}\left(-\frac{\pi}{4}\right)$

c $\frac{1}{z} = \text{cis}\left(-\frac{\pi}{4}\right)$

d $-iz = \text{cis}\left(-\frac{\pi}{4}\right)$

Solutions to multiple-choice questions

1 **C**
$$\begin{aligned}\frac{1}{2-u} &= \frac{1}{1-i}\\ &= \frac{1}{1-i} \times \frac{1+i}{1+i}\\ &= \frac{1+i}{2}\\ &= \frac{1}{2} + \frac{1}{2}i\end{aligned}$$

2 **D** $i = \text{ cis } \frac{\pi}{2}$, so the point will be rotated by $\frac{\pi}{2}$.

3 **C**
$$\begin{aligned}|z| &= 5\\ \left|\frac{1}{z}\right| &= \frac{1}{|z|}\\ &= \frac{1}{5}\end{aligned}$$

4 **D**
$$\begin{aligned}(x+yi)^2 &= x^2 + 2xyi + y^2i^2\\ &= (x^2 - y^2) + 2xyi\end{aligned}$$

Therefore

$x^2 - y^2 = 0$ and $2xy = -32$.

Therefore

$x^2 - y^2 = 0 \Rightarrow y = \pm x$

If $y = x$ then

$2xy = -32$

has no solution. If $y = -x$, then

$$\begin{aligned} 2xy &= -32 \\ -2x^2 &= -32 \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

Therefore, $x = 4, y = -4$ or $x = 4, y = -4$.

5 **D** Completing the square gives,

$$\begin{aligned} z^2 + 6z + 10 &= z^2 + 6z + 9 + 1 \\ &= (z+3)^2 + 1 \\ &= (z+3)^2 - i^2 \\ &= (z+3-i)(z+3+i). \end{aligned}$$

6 **E** Completing the square gives,

$$\begin{aligned} \frac{1}{1-i} &= \frac{1}{1-i}\frac{1+i}{1+i} \\ &= \frac{1+i}{2} \\ &= \frac{1}{2} + \frac{1}{2}i \end{aligned}$$

Therefore,

$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

and

$$\theta = \frac{\pi}{4}.$$

7 **D**

$$\begin{aligned} \frac{z-2i}{z-(3-2i)} &= 2 \\ z - 2i &= 2(z-(3-2i)) \\ z - 2i &= 2z - 2(3-2i) \\ z &= 2(3-2i) - 2i \\ &= 6 - 6i \end{aligned}$$

8 **D**

$$\begin{aligned} z^2(1+i) &= 2 - 2i \\ z^2 &= \frac{2-2i}{1+i} \\ &= \frac{(2-2i)(1-i)}{2} \\ &= (1-i)^2 \\ &= (-1+i)^2 \end{aligned}$$

9 **B**

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (8i)^2 - 4(2+2i)(-4(1-i)) \\ &= 64i^2 + 16(2+2i)(1-i) \\ &= -64 + 32(1+i)(1-i) \\ &= -64 + 32(1-i^2) \\ &= -64 + 32 \times 2 \\ &= 0 \end{aligned}$$

10 **D** $\text{Arg}(1+ai) = \frac{\pi}{6}$

$$\tan^{-1} a = \frac{\pi}{6}$$

$$a = \tan\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{\sqrt{3}}$$

Solutions to extended-response questions

1 **a** $z^2 - 2\sqrt{3}z + 4 = 0$

Completing the square gives

$$z^2 - 2\sqrt{3}z + 3 + 1 = 0$$

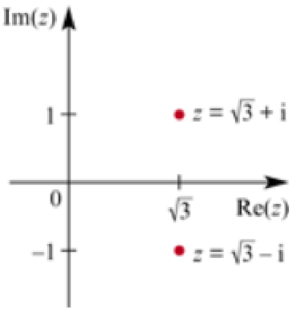
$$\therefore\ (z-\sqrt{3})^2 + 1 = 0$$

$$\therefore\ (z-\sqrt{3})^2 - i^2 = 0$$

$$\therefore\ (z-\sqrt{3}+i)(z-\sqrt{3}-i) = 0$$

$$\therefore\ z = \sqrt{3} \pm i$$

b **i**



ii $|\sqrt{3}+i| = |\sqrt{3}-i| = 2$

The circle has centre the origin and radius 2.

The cartesian equation is $x^2 + y^2 = 4$.

iii The circle passes through $(0, 2)$ and $(0, -2)$. The corresponding complex numbers are$2i$ and $-2i$. So $a = 2$

2 $|z| = 6$

a **i** $|(1+i)z| = |1+i||z|$

$$= \sqrt{2} \times 6$$

$$= 6\sqrt{2}$$

ii $|(1+i)z - z| = |z + iz - z|$

$$= |iz|$$

$$= |i|\,|z|$$

$$= 6$$

b A is the point corresponding to z, and $|OA| = 6$.

B is the point corresponding to $(1+i)z$, and $|OB| = 6\sqrt{2}$.

From part **b**, $|AB| = |(1+i)z - z|$

$$= 6$$

Therefore ΔOAB is isosceles.

Note also that

$$|OA|^2 + |AB|^2 = 6^2 + 6^2 = 72$$

$$\text{and} \quad |OB|^2 = (6\sqrt{2})^2$$

$$= 72$$

The converse of Pythagoras' theorem gives the triangle is right-angled at A.

3 $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

a $1 + z = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

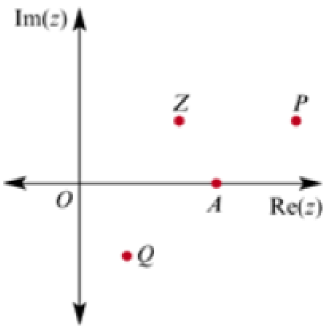
$$= \left(1 + \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}i$$

$$= \frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

and $1 - z = 1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

$$= \left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}i$$

$$= \frac{\sqrt{2}-1}{\sqrt{2}} \frac{1}{\sqrt{2}}i$$



b $|OP|^2 = \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$

$$= \frac{1}{2}(2 + 2\sqrt{2} + 1 + 1)$$

$$= 2 + \sqrt{2}$$

$$|OQ|^2 = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2}(2 - 2\sqrt{2} + 1 + 1)$$

$$= 2 - \sqrt{2}$$

$$|QP| = |-1 + z + 1 + z|$$

$$= |2z|$$

$$= 2|z|$$

$$= 2$$

and $|QP|^2 = 4$

Therefore $|QP|^2 = |OP|^2 + |OQ|^2$

By the converse of Pythagoras' theorem $\angle POQ$ is a right angle, i.e. $\angle POQ = \frac{\pi}{2}$

Now $\frac{|OP|}{|OQ|} = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$

$$= \sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}} \times \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$$

$$= \frac{2+\sqrt{2}}{\sqrt{2}}$$

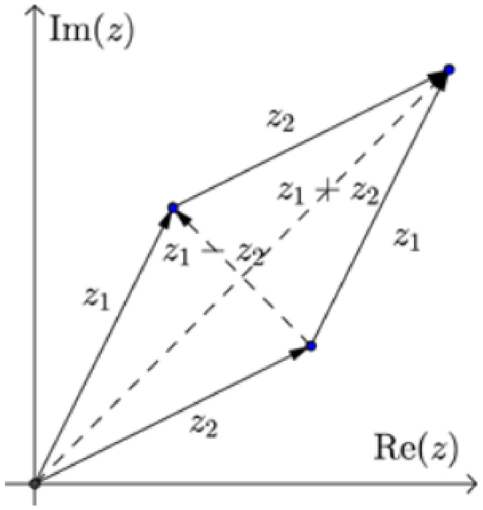
$$= \sqrt{2} + 1$$

4 For this question we will use the fact that $|z|^2 = z\overline{z}$. This is easy to prove.

a
$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\ &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2 \\ &= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2 \end{aligned}$$

b
$$\begin{aligned} |z_1 - z_2|^2 &= (z_1 - z_2)(\overline{z_1 - z_2}) \\ &= (z_1 - z_2)(\overline{z_1} - \overline{z_2}) \\ &= z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - \overline{z_1}z_2 \\ &= |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2) \end{aligned}$$

c Since
$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$$
and
$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$$
we can add these two equations to give,
$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.



This result has a geometric interpretation. By interpreting complex numbers z_1 and z_2 as vectors, we obtain a parallelogram with diagonals whose vectors are $z_1 + z_2$ and $z_1 - z_2$. This result then shows that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals

5 a For this question we will use the fact that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$. This is easy to prove if you haven't already seen it done.

i
$$\begin{aligned} \overline{\overline{z_1} z_2} &= \overline{\overline{z_1}}\ \overline{z_2} \\ &= z_1\overline{z_2} \end{aligned}$$

ii First note that $z + \overline{z} = 2\mathrm{Re}\,(z)$. Now using part (i) we have
$$\begin{aligned} z_1\overline{z_2} + \overline{z_1}z_2 &= \overline{\overline{z_1}z_2} + \overline{z_1}z_2 \\ &= 2\mathrm{Re}\,(\overline{z_1}z_2), \end{aligned}$$
which is a real number.

iii First note that $z - \overline{z} = 2i\,\mathrm{Im}\,(z)$. Now using part (i) we have
$$\begin{aligned} z_1\overline{z_2} - \overline{z_1}z_2 &= \overline{\overline{z_1}z_2} - \overline{z_1}z_2 \\ &= 2i\,\mathrm{Im}\,(\overline{z_1}z_2), \end{aligned}$$
which is an imaginary number.

iv Adding the results of the two previous questions gives

$$\begin{aligned}(z_1\overline{z_2}+\overline{z_1}z_2)^2+(z_1\overline{z_2}-\overline{z_1}z_2)^2 &= (2\text{Re}\,(\overline{z_1}z_2))^2-(2i\,\text{Im}\,(\overline{z_1}z_2))^2\\ &= 4(\text{Re}\,(\overline{z_1}z_2))^2+4(\text{Im}\,(\overline{z_1}z_2))^2\\ &= 4((\text{Re}\,(\overline{z_1}z_2))^2+(\text{Im}\,(\overline{z_1}z_2))^2)\\ &= 4|\overline{z_1}z_2|^2\\ &= 4|\overline{z_1}||z_2|^2\\ &= 4|z_1||z_2|^2\\ &= 4|z_1z_2|^2.\end{aligned}$$

b

$$\begin{aligned}(|z_1|+|z_2|)^2-|z_1+z_2|^2 &= |z_1|^2+2|z_1||z_2|+|z_2|^2-(z_1+z_2)\overline{(z_1+z_2)}\\ &= |z_1|^2+2|z_1||z_2|+|z_2|^2-(z_1+z_2)(\overline{z_1}+\overline{z_2})\\ &= |z_1|^2+2|z_1||z_2|+|z_2|^2-(z_1\overline{z_1}+z_2\overline{z_2}+z_1\overline{z_2}+\overline{z_1}z_2)\\ &= |z_1|^2+2|z_1||z_2|+|z_2|^2-(|z_1|^2+|z_2|^2+z_1\overline{z_2}+\overline{z_1}z_2)\\ &= |z_1|^2+2|z_1||z_2|+|z_2|^2-|z_1|^2-|z_2|^2-(z_1\overline{z_2}+\overline{z_1}z_2)\\ &= 2|z_1||z_2|-(z_1\overline{z_2}+\overline{z_1}z_2)\\ &= 2|z_1||z_2|-2\text{Re}\,(\overline{z_1}z_2)\\ &= 2|\overline{z_1}||z_2|-2\text{Re}\,(\overline{z_1}z_2)\\ &= 2|\overline{z_1}z_2|-2\text{Re}\,(\overline{z_1}z_2)\\ &\geq 0\end{aligned}$$

c This question simply requires a trick:
$|z_1|=|(z_1-z_2)+z_2|\leq|z_1-z_2|+|z_2|$.

Therefore,

$|z_1-z_2|\geq|z_1|-|z_2|$.

6 $z=\text{cis}\theta$

a

$$\begin{aligned}z+1 &= \text{cis}\theta+1\\ &= \cos\theta+i\sin\theta+1\\ &= (1+\cos\theta)+i\sin\theta\\ |z+1| &= \sqrt{(1+\cos\theta)^2+\sin^2\theta}\\ &= \sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\\ &= \sqrt{1+2\cos\theta+1}\\ &= \sqrt{2+2\cos\theta}\\ &= \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\\ &= 2\cos\left(\frac{\theta}{2}\right)\text{ since }0\leq\frac{\theta}{2}\leq\frac{\pi}{2}.\end{aligned}$$

To find the argument, we find that

$$\begin{aligned}\frac{\sin\theta}{1+\cos\theta} &= \frac{\sin\theta}{2\cos^2\frac{\theta}{2}}\\ &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\\ &= \frac{\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}\\ &= \tan\frac{\theta}{2}\end{aligned}$$

so that $\operatorname{Arg}(z+1)=\frac{\theta}{2}$.

b

$$\begin{aligned}z-1 &= \operatorname{cis}\theta-1\\ &= \cos\theta+i\sin\theta-1\\ &= (\cos\theta-1)+i\sin\theta\\ |z-1| &= \sqrt{(\cos\theta-1)^2+\sin^2\theta}\\ &= \sqrt{\cos^2\theta-2\cos\theta+1+\sin^2\theta}\\ &= \sqrt{2-2\cos\theta}\\ &= \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\\ &= 2\sin\left(\frac{\theta}{2}\right)\text{ since }0\le\frac{\theta}{2}\le\frac{\pi}{2}.\end{aligned}$$

To find the argument, we evaluate

$$\begin{aligned}\frac{\sin\theta}{\cos\theta-1} &= -\frac{\sin\theta}{1-\cos\theta}\\ &= -\frac{\sin\theta}{2\sin^2\left(\frac{\theta}{2}\right)}\\ &= -\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}\\ &= -\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\\ &= -\cot\left(\frac{\theta}{2}\right)\\ &= \tan\left(\frac{\theta}{2}+\frac{\pi}{2}\right)\end{aligned}$$

so that $\operatorname{Arg}(z-1)=\frac{\pi}{2}+\frac{\theta}{2}$.

c

$$\left|\frac{z-1}{z+1}\right| = \frac{|z-1|}{|z+1|}$$
$$= \frac{2\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)}$$
$$= \tan\left(\frac{\theta}{2}\right)$$

$$\text{Arg}\left(\frac{z-1}{z+1}\right) = \text{Arg}(z-1) - \text{Arg}(z+1)$$
$$= \frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}$$
$$= \frac{\pi}{2}$$

7 a $\Delta = b^2 - 4ac$

b The equation has no real solutions if and only if $b^2 - 4ac < 0$.

c If $b^2 - 4ac$ then we can assume that

$$z_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } z_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a}.$$

It follows that P_1 has coordinates

$$\left(\frac{-b}{2a}, \frac{\sqrt{4ac - b^2}}{2a}\right)$$

and P_2 has coordinates

$$\left(\frac{-b}{2a}, -\frac{\sqrt{4ac - b^2}}{2a}\right).$$

i

$$z_1 + z_2 = -\frac{b}{a}$$
$$|z_1| = |z_2| = \sqrt{\left(\frac{-b}{2a}\right)^2 + \left(\frac{\sqrt{4ac - b^2}}{2a}\right)^2}$$
$$= \sqrt{\frac{b^2}{4a^2} + \frac{4ac - b^2}{4a^2}}$$
$$= \sqrt{\frac{c}{a}}$$

ii

To find $\angle P_1OP_2$ it will also help to find

$$z_1 - z_2 = \frac{i\sqrt{4ac - b^2}}{a}$$
$$|z_1 - z_2| = \frac{\sqrt{4ac - b^2}}{|a|}$$

Therefore, with reference to the diagram below, we use the cosine law toshow that

$$P_1P_2 = OP_1^2 + OP_2^2 - 2 \cdot OP_1 \cdot OP_2 \cdot \cos\theta$$

$$\frac{4ac-b^2}{a^2} = \frac{c}{a} + \frac{c}{a} - 2\frac{c}{a}\cos\theta$$

$$\frac{4ac-b^2}{a^2} = \frac{2c}{a} - 2\frac{c}{a}\cos\theta$$

$$\frac{4ac-b^2}{a^2} = \frac{2c}{a}(1-\cos\theta)$$

$$\frac{4ac-b^2}{a} = 2c(1-\cos\theta)$$

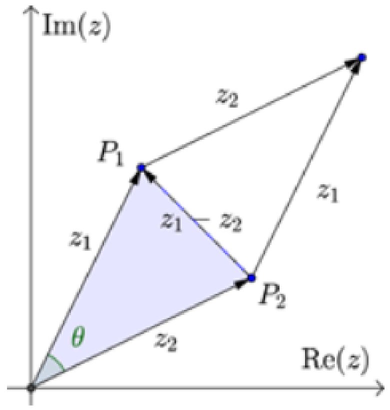
$$1-\cos\theta = \frac{4ac-b^2}{2ac}$$

$$\cos\theta = 1 - \frac{4ac-b^2}{2ac}$$

$$\cos\theta = \frac{b^2-2ac}{2ac.}$$

Therefore

$$\cos(\angle P_1OP_2) = \frac{b^2-2ac}{2ac}.$$



8 a It's perhaps fastest to simply use the quadratic formula here:

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2-4(1)(1)}}{2} \\ &= \frac{-1 \pm \sqrt{-3}}{2} \\ &= \frac{-1 \pm i\sqrt{3}}{2} \end{aligned}$$

so that

$$z_1 = \frac{-1+i\sqrt{3}}{2} \text{ and } z_2 = \frac{-1-i\sqrt{3}}{2}.$$

b We prove the first equality. The proof for the second is similar. We have

$$\begin{aligned} z_2^2 &= \left(\frac{-1-i\sqrt{3}}{2}\right)^2 \\ &= \frac{1}{4}(1+i\sqrt{3})^2 \\ &= \frac{1}{4}(1+2i\sqrt{3}+3i^2) \\ &= \frac{1}{4}(-2+2i\sqrt{3}) \\ &= \frac{-1+i\sqrt{3}}{2} \\ &= z_1, \end{aligned}$$

as required.

c First consider $z_1 = \dfrac{-1+i\sqrt{3}}{2}$. The point is in the second quadrant.

$$\begin{aligned} r &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)} \\ &= \sqrt{\frac{1}{4}+\frac{3}{4}} \\ &= 1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3} \end{aligned}$$

$$\therefore \quad \frac{-1+i\sqrt{3}}{2} = 1\ \text{cis}\left(\frac{2\pi}{3}\right).$$

Now consider $z_2 = \dfrac{-1-i\sqrt{3}}{2}$. The point is in the third quadrant.

$$\begin{aligned} r &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)} \\ &= \sqrt{\frac{1}{4}+\frac{3}{4}} \\ &= 1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= -\frac{2\pi}{3} \end{aligned}$$

$$\therefore \quad \frac{-1-i\sqrt{3}}{2} = 1\ \text{cis}\left(-\frac{2\pi}{3}\right).$$

d Plot points O, P_1 and P_2. From this, you will see that

$$\begin{aligned} A &= \frac{bh}{2} \\ &= \frac{\sqrt{3}\times\frac{1}{2}}{2} \\ &= \frac{\sqrt{3}}{4}. \end{aligned}$$